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Multiple Positive Solutions for Nonlinear Second-Order m-Point Impulsive Boundary Value Problems on Time Scales

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Abstract. In this paper, by using fixed point index theory, we study the existence of positive solutions for nonlinear second-order m-point impulsive boundary value problems on time scales. As an application, we give an example to demonstrate our results.

1. Introduction

The theory of impulsive differential equations describe processes which experience a sudden change of their state at certain moments. Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For the introduction of the basic theory of impulsive equations, see [1, 2, 11, 16] and the references therein.

The study of dynamic equations on time scales goes back to Stefan Hilger [9]. Now it is still a new area of fairly therotical exploration in Mathematics. We refer to the books by Bohner and Peterson [4, 5].

Recently, the existence and multiplicity of positive solutions for linear and nonlinear second-order impulsive differential equations have been studied extensively [6, 7, 10, 20]. However, the corresponding theory of such equations is still in the beginning stages of its development, especially the impulsive dynamic system on time scales, see [3, 8, 13, 17, 18]. There is not much work on second-order with m-point impulsive boundary value problems on time scales, see [14, 19].

In [7], Guo studied the following two-point boundary value problem

$$\begin{cases} -x'' = f(t,x,x'), \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)x'(t_k)), \quad (k = 1,2,\dots,m), \\ ax(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_0^*. \end{cases}$$

Utilizing the Darbo fixed point theorem, Guo obtained the existence criteria of at least one solution.

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In [10], Hu, Liu and Wu studied second-order two-point impulsive boundary value problem

$$\begin{cases} -u'' = h(t)f(t,u), \quad t \in J', \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), \\ \Delta u|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1,2,...,m, \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

By using the fixed point theorem in cone, they obtained the existence criteria of one or two positive solutions.

In [15], Ma considered the existence and multiplicity of positive solutions for the m-boundary value problems

$$\begin{cases} (p(t)u')' - q(t)u + f(t,u) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases}$$

The main tool is Guo-Krasnoselskii fixed point theorem.

In [14], Li, Chen and Wu studied the following m-point boundary value problems for p-Laplacian impulsive dynamic equations on time scales

$$\begin{cases} [\phi_p(y^{\triangle}(t))]^{\nabla} + w(t)f(t,y(t)) = 0, \ t \in [0,T]_{\mathbb{T}}, \ t \neq t_k, \ k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(u(t_k)), \\ y(0) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y^{\triangle}(T) = 0, \end{cases}$$

In accordance with the Leray-Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder type, they get the existence of at least one positive solution. They also considered the existence of at least three positive solutions by using a new fixed point theorem.

Motivated by the above results, in this study, we consider the following second-order m-point impulsive boundary value problem (BVP) on time scales

$$\begin{cases} u^{\triangle\triangle}(t) + q(t)f(t,u(t)) = 0, \ t \in J := [0,1]_{\mathbb{T}}, \ t \neq t_k, \ k = 1,2,...,n, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \\ \Delta u^{\triangle}|_{t=t_k} = -J_k(u(t_k)), \\ au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + du^{\triangle}(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0,1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $t_k \in (0,1)_{\mathbb{T}}$, $k = 1,2,...,n$ with $0 < t_1 < t_2 < ... < t_n < 1$. $\Delta u|_{t=t_k}$ and $\Delta u^{\triangle}|_{t=t_k}$ denote the jump of $u(t)$ and $u^{\triangle}(t)$ at $t = t_k$, i.e.,

$$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), \ \ \Delta u^{\triangle}|_{t=t_k} = u^{\triangle}(t_k^+) - u^{\triangle}(t_k^-),$$

where $u(t_k^+), u^{\triangle}(t_k^+)$ and $u(t_k^-), u^{\triangle}(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ and $u^{\triangle}(t)$ at $t = t_k$, $k = 1,2,...,n$, respectively.

Throughout this paper we assume that following conditions hold:

$(H1)$ $a,b,c,d \in [0,\infty)$ with $ac + ad + bc > 0$; $\alpha_i, \beta_i \in [0,\infty)$, $\xi_i \in (0,1)_{\mathbb{T}}$ for $i \in \{1,2,...,m-2\}$,

($H2$) $f \in C([0,1]_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}^+)$, $q \in C([0,1]_{\mathbb{T}}, \mathbb{R}^+)$,

($H3$) $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function, $J_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $(c(1-t_k)+d)J_k(u(t_k)) > cI_k(u(t_k))$, $k = 1,2,...,n$.

By means of the fixed point index theory in the cone [12], we get the existence of at least two positive solutions for the impulsive BVP (1.1). Then we generalized this to obtain many positive solutions. In fact, our results is also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the results can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2. Preliminaries

In this section, to state the main results of this paper, we need the following lemmas.

Throughout the rest of this paper, we assume that the points of impulse t_k are right dense for each $k = 1,2,...,n$. Let $J = [0,1]_{\mathbb{T}}$, $J' = J \setminus \{t_1, t_2, ..., t_n\}$. We define

$$\begin{aligned} \mathbb{B} \;=\; & \{u \mid u : [0,1]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \text{ and there exist} \\ & u(t_k^-) \text{ and } u(t_k^+) \text{ with } u(t_k^-) = u(t_k) \text{ for } k = 1,2,...,n\}. \end{aligned}$$

Then $\mathbb{B}$ is a real Banach space with the norm $\|u\| = \sup_{t\in[0,1]_{\mathbb{T}}} |u(t)|$. A function $u \in \mathbb{B} \cap C^2(J')$ is called a solution to (1.1) if it satisfies all equations of (1.1).

$$\begin{aligned} K \;=\; & \Big\{u \in \mathbb{B} : u(t) \text{ is nonnegative, nondecreasing on } [0,1]_{\mathbb{T}} \text{ and } u^{\triangle}(t) \text{ is nonincreasing on } [0,1]_{\mathbb{T}}, \\ & au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)\Big\}. \end{aligned}$$

Obviously, K is a cone in $\mathbb{B}$. We note that, for each $u \in K$, $\|u\| = \sup_{t\in[0,1]_{\mathbb{T}}} |u(t)| = u(1)$.

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$u^{\triangle\triangle}(t) = 0,\; t \in J := [0,1]_{\mathbb{T}},\; t \neq t_k,\; k = 1,2,...,n, \tag{2.1}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, \;\; \theta^{\Delta}(0) = a, \\ \varphi(1) = d, \;\; \varphi^{\Delta}(1) = -c. \end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for θ and φ the following equations:

$$\theta(t) = b + at, \;\; \varphi(t) = d + c(1-t). \tag{2.3}$$

Set

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1-\xi_i)) \\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) & -\sum_{i=1}^{m-2} \beta_i (d + c(1-\xi_i)) \end{vmatrix}, \tag{2.4}$$

and

$$\rho := ad + ac + bc. \tag{2.5}$$

Lemma 2.1. *Let* $(H1)-(H3)$ *hold. Assume that*

$(H4)$ $\Delta \neq 0$.

If $u \in \mathbb{B} \cap C^2(J')$ *is a solution of the equation*

$$u(t) = \int_0^1 G(t,\sigma(s))\, q(s) f(s,u(s))\, \triangle s + \sum_{k=1}^{n} W_k(t,t_k) + A(f)\theta(t) + B(f)\varphi(t), \tag{2.6}$$

where

$$W_k(t,t_k) = \frac{1}{\rho}\begin{cases} (b+at)\left(-cI_k(u(t_k)) + (d+c(1-t_k))J_k(u(t_k))\right), & t<t_k, \\ (d+c(1-t))\left(aI_k(u(t_k)) + (b+at_k)J_k(u(t_k))\right), & t_k \le t, \end{cases} \tag{2.7}$$

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b+as)(d+c(1-t)), & \sigma(s) \le t, \\ (b+at)(d+c(1-s)), & t \le s, \end{cases} \tag{2.8}$$

$$A(f) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i \mathcal{K}_i & \rho - \sum_{i=1}^{m-2}\alpha_i(d+c(1-\xi_i)) \\ \sum_{i=1}^{m-2}\beta_i \mathcal{K}_i & -\sum_{i=1}^{m-2}\beta_i(d+c(1-\xi_i)) \end{vmatrix}, \tag{2.9}$$

$$B(f) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i(b+a\xi_i) & \sum_{i=1}^{m-2}\alpha_i \mathcal{K}_i \\ \rho - \sum_{i=1}^{m-2}\beta_i(b+a\xi_i) & \sum_{i=1}^{m-2}\beta_i \mathcal{K}_i \end{vmatrix}, \tag{2.10}$$

and

$$\mathcal{K}_i := \int_0^1 G(\xi_i,\sigma(s))\, q(s) f(s,u(s))\, \triangle s + \sum_{k=1}^{n} W_k(\xi_i,t_k), \tag{2.11}$$

then u *is a solution of the impulsive boundary value problem* (1.1).

Proof. Let u satisfies the integral equation (2.6), then we have

$$u(t) = \int_0^1 G(t,\sigma(s))\, q(s) f(s,u(s))\, \triangle s + \sum_{k=1}^{n} W_k(t,t_k) + A(f)\theta(t) + B(f)\varphi(t),$$

i.e.,

$$\begin{aligned} u(t) &= \int_0^t \frac{1}{\rho}(b+a(\sigma(s))(d+c(1-t))q(s)f(s,u(s))\,\triangle s + \int_t^1 \frac{1}{\rho}(b+at)(d+c(1-\sigma(s)))q(s)f(s,u(s))\,\triangle s \\ &\quad + \sum_{0<t_k<t}(d+c(1-t))\left(aI_k(u(t_k)) + (b+at_k)J_k(u(t_k))\right) \\ &\quad + \sum_{t<t_k<1}(b+at)\left(-cI_k(u(t_k)) + (d+c(1-t_k))J_k(u(t_k))\right) + A(f)(b+at) + B(f)(d+c(1-t)), \\ u^{\triangle}(t) &= -\int_0^t \frac{c}{\rho}(b+a(\sigma(s)))q(s)f(s,u(s))\,\triangle s + \int_t^1 \frac{a}{\rho}(d+c(1-\sigma(s)))q(s)f(s,u(s))\,\triangle s \\ &\quad - \sum_{0<t_k<t} c\left(aI_k(u(t_k)) + (b+at_k)J_k(u(t_k))\right) \\ &\quad + \sum_{t<t_k<1} a\left(-cI_k(u(t_k)) + (d+c(1-t_k))J_k(u(t_k))\right) + A(f)a - B(f)c. \end{aligned}$$

So that

$$\begin{aligned} u^{\triangle\triangle}(t) &= \frac{1}{\rho}\left(-c(b+a(\sigma(t))) - a(d+c(1-\sigma(t)))\right) q(t) f\left(t, u(t)\right) \\ &= \frac{1}{\rho}\left(-(ad+ac+bc)\right) q(t) f\left(t, u(t)\right) = -q(t) f\left(t, u(t)\right), \end{aligned}$$

$$u^{\triangle\triangle}(t) + q(t) f(t, u(t)) = 0.$$

Since

$$\begin{aligned} u(0) &= \int_0^1 \frac{b}{\rho}(d + c(1-\sigma(s)))q(s) f\left(s, u(s)\right) \triangle s \\ &\quad + \sum_{k=1}^{n} b\left(-cI_k(u(t_k)) + (d + c(1-t_k))J_k\left(u(t_k)\right)\right) + A(f)b + B(f)(d+c), \end{aligned}$$

$$\begin{aligned} u^{\triangle}(0) &= \int_0^1 \frac{a}{\rho}(d + c(1-\sigma(s)))q(s) f\left(s, u(s)\right) \triangle s \\ &\quad + \sum_{k=1}^{n} a\left(-cI_k(u(t_k)) + (d + c(1-t_k))J_k\left(u(t_k)\right)\right) + A(f)a - B(f)c, \end{aligned}$$

we have that

$$au(0) - bu^{\triangle}(0) = B(f)\left(ad + ac + bc\right)$$

$$= \sum_{i=1}^{m-2} \alpha_i \left[\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s) f\left(s, u(s)\right) \triangle s + \sum_{k=1}^{n} W_k\left(\xi_j, t_k\right) + A(f)(b + a\xi_j) + B(f)(d + c(1-\xi_j)) \right]. \tag{2.12}$$

Since

$$\begin{aligned} u(1) &= \int_0^1 \frac{d}{\rho}(b + a(\sigma(s))q(s) f\left(s, u(s)\right) \triangle s \\ &\quad + \sum_{k=1}^{n} d\left(aI_k(u(t_k)) + (b + at_k)J_k\left(u(t_k)\right)\right) + A(f)(b+a) + B(f)d, \\ u^{\triangle}(1) &= -\int_0^1 \frac{c}{\rho}(b + a(\sigma(s)))q(s) f\left(s, u(s)\right) \triangle s \\ &\quad - \sum_{k=1}^{n} c\left(aI_k(u(t_k)) + (b + at_k)J_k\left(u(t_k)\right)\right) + A(f)a - B(f)c, \end{aligned}$$

we have that

$$cu(1) + du^{\triangle}(1) = A(f)\left(ad + ac + bc\right)$$

$$= \sum_{i=1}^{m-2} \beta_i \left[\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s) f\left(s, u(s)\right) \triangle s + \sum_{k=1}^{n} W_k\left(\xi_i, t_k\right) + A(f)(b + a\xi_i) + B(f)(d + c(1-\xi_i)) \right]. \tag{2.13}$$

From (2.5), (2.12) and (2.13), we get that

$$\begin{cases} \left[-\sum_{i=1}^{m-2}\alpha_i(b+a\xi_i)\right]A(f)+\left[\rho-\sum_{i=1}^{m-2}\alpha_i(d+c(1-\xi_i)\right]B(f)=\sum_{i=1}^{m-2}\alpha_i\mathcal{K}_i \\ \left[\rho-\sum_{i=1}^{m-2}\beta_i(b+a\xi_i)\right]A(f)+\left[-\sum_{i=1}^{m-2}\beta_i(d+c(1-\xi_i)\right]B(f)=\sum_{i=1}^{m-2}\beta_i\mathcal{K}_i \end{cases},$$

which implies that $A(f)$ and $B(f)$ satisfy (2.9) and (2.10), respectively. □

Lemma 2.2. *Let* $(H1)-(H3)$ *hold. Assume*

$(H5)$ $\Delta<0,\ \rho-\sum_{i=1}^{m-2}\beta_i(b+a\xi_i)>0,\ a-\sum_{i=1}^{m-2}\alpha_i>0.$

Then for $u\in\mathbb{B}\cap C^2(J')$ *with* $f,q\geq 0$, *the solution* u *of the problem* (1.1) *satisfies*

$u(t)\geq 0$ *for* $t\in[0,1]_{\mathbb{T}}$.

Proof. It is an immediate subsequence of the facts that $G\geq 0$ on $[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}$ and $A(f)\geq 0,\ B(f)\geq 0$. □

Lemma 2.3. *Let* $(H1)-(H3)$ *and* $(H5)$ *hold. Assume*

$(H6)$ $c-\sum_{i=1}^{m-2}\beta_i<0.$

Then the solution $u\in\mathbb{B}\cap C^2(J')$ *of the problem* (1.1) *satisfies* $u^{\triangle}(t)\geq 0$ *for* $t\in[0,1]_{\mathbb{T}}$.

Proof. Assume that the inequality $u^{\triangle}(t)<0$ holds. Since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one can verify that

$u^{\triangle}(1)\leq u^{\triangle}(t),\ t\in[0,1]_{\mathbb{T}}$.

From the boundary conditions of the problem (1.1), we have

$$-\frac{c}{d}u(1)+\frac{1}{d}\sum_{i=1}^{m-2}\beta_i u(\xi_i)\leq u^{\triangle}(t)<0.$$

The last inequality yields

$$-cu(1)+\sum_{i=1}^{m-2}\beta_i u(\xi_i)<0.$$

Therefore, we obtain that

$$\sum_{i=1}^{m-2}\beta_i u(1)<\sum_{i=1}^{m-2}\beta_i u(\xi_i)<cu(1),$$

i.e.,

$$\left(c-\sum_{i=1}^{m-2}\beta_i\right)u(1)>0.$$

According to Lemma 2.2, we have that $u(1)\geq 0$. So, $c-\sum_{i=1}^{m-2}\beta_i>0$. However, this contradicts to condition $(H6)$. Consequently, $u^{\triangle}(t)\geq 0$ for $t\in[0,1]_{\mathbb{T}}$. □

Lemma 2.4. *If* $(H1)-(H6)$ *hold, then* $\min_{t\in[0,1]_{\mathbb{T}}} u(t) \geq \gamma\|u\|$ *for* $u \in K$, *where* $\gamma = \dfrac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{a-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}$.

Proof. Since $u \in K$, nonnegative and nondecreasing

$$\|u\| = u(1), \quad \min_{t\in[0,1]_{\mathbb{T}}} u(t) = u(0).$$

On the other hand, $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$. So, for every $t \in [0,1]_{\mathbb{T}}$, we have

$$\frac{u(t)-u(0)}{t} \geq \frac{u(1)-u(0)}{1},$$

i.e., $u(t) \geq (1-t)u(0) + tu(1)$. Therefore,

$$\sum_{i=1}^{m-2}\alpha_i u(\xi_i) \geq \sum_{i=1}^{m-2}\alpha_i(1-\xi_i)u(0) + \sum_{i=1}^{m-2}\alpha_i\xi_i u(1).$$

This together with $au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$, implies that

$$u(0) \geq \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{a-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}u(1).$$

So, the proof of Lemma is completed. □

Now define an operator $T : K \longrightarrow \mathbb{B}$ by

$$Tu(t) = \int_0^1 G(t,\sigma(s))\,q(s)f(s,u(s))\,\triangle s + \sum_{k=1}^{n} W_k(t,t_k) + A(f)\theta(t) + B(f)\varphi(t), \tag{2.14}$$

where W_k, G, $A(f)$, $B(f)$ and θ, φ are defined as in (2.7), (2.8), (2.9), (2.10) and (2.3) respectively.

Lemma 2.5. *Let* $(H1)-(H6)$ *hold. Then* $T : K \to K$ *is completely continuous.*

Proof. By Arzela-Ascoli theorem, we can easily prove that operator T is completely continuous. □

3. Main Results

The following fixed point theorem is fundamental and important to the proofs of our main results.

Lemma 3.1. *(See [12]). Let* K *be a cone in a real Banach space* $\mathbb{B}$. *Let* D *be an open bounded subset of* $\mathbb{B}$ *with* $D_K = D \cap K \neq \emptyset$ *and* $\overline{D}_K \neq K$. *assume that* $T : \overline{D}_K \to K$ *is completely continuous such that* $x \neq Tx$ *for* $x \in \partial D_K$. *Then the following results hold:*

(*i*) *If* $\|Tx\| \leq x$, $x \in \partial D_K$, *then* $i_K(T, D_K) = 1$.

(*ii*) *If there exists* $e \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial D_K$ *and all* $\lambda > 0$, *then* $i_K(T, D_K) = 0$.

(*iii*) *Let U be open in K such that $\overline{U} \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(T, U_K) = 0$, then T has a fixed point in $D_K \setminus \overline{U}_K$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, U_K) = 1$.*

Now we consider the existence of at least two positive solutions for the BVP (1.1) by the fixed point theorem in [12].

We define

$$\begin{aligned} K_\rho &= \{u \in K : \|u\| < \rho\}, \\ \Omega_\rho &= \{u \in K : \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\} = \{u \in K : \gamma\|u\| \leq \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\}. \end{aligned}$$

Lemma 3.2. *Ω_ρ has the following properties:*

(a) Ω_ρ is open relative to K.

(b) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.

(c) $u \in \Omega_\rho$ if and only if $\min_{t\in[0,1]} u(t) = \gamma\rho$.

(d) If $u \in \Omega_\rho$, then $\gamma\rho \leq u(t) \leq \rho$ for $t \in [0, 1]$.

Now for convenience we introduce the following notations. Let

$$A = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) & \rho - \sum_{i=1}^{m-2}\alpha_i(d + c(1-\xi_i)) \\ \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) & -\sum_{i=1}^{m-2}\beta_i(d + c(1-\xi_i)) \end{vmatrix},$$

$$B = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}\alpha_i(b + a\xi_i) & \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) \\ \rho - \sum_{i=1}^{m-2}\beta_i(b + a\xi_i) & \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G\left(\xi_i, \sigma(s)\right) q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) \end{vmatrix},$$

$$\begin{aligned} D &= \left[\int_0^1 G\left(\sigma(s), \sigma(s)\right) q(s)\triangle s + \frac{n}{\rho}(c+d)(2a+b) + A(b+a) + B(d+c)\right]^{-1}, \\ D^* &= \left[\frac{d}{c+d}\int_0^1 G\left(\sigma(s), \sigma(s)\right) q(s)\triangle s\right]^{-1}, \\ H^\rho_{\gamma\rho} &= \min\left\{\min_{t\in[0,1]} f(t,u) : u \in [\gamma\rho, \rho]\right\}, \\ F^\rho_0 &= \max\left\{\max_{t\in[0,1]} f(t,u) : u \in [0, \rho]\right\}, \\ I^\rho_0(k) &= \max\left\{I_k(u) : u \in [0, \rho]\right\},\ k = 1, 2, ..., n, \\ J^\rho_0(k) &= \max\left\{J_k(u) : u \in [0, \rho]\right\},\ k = 1, 2, ..., n. \end{aligned}$$

Theorem 3.3. *Suppose $(H1) - (H6)$ hold.*

$(H7)$ There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that

$$F^{\rho_1}_0 < \rho_1 D,\ I^{\rho_1}_0(k) < \rho_1 D,\ J^{\rho_1}_0(k) < \rho_1 D,\ H^{\rho_2}_{\gamma\rho_2} > \rho_2 D^*,\ F^{\rho_3}_0 < \rho_3 D,\ I^{\rho_3}_0(k) < \rho_3 D,\ J^{\rho_3}_0(k) < \rho_3 D.$$

Then problem (1.1) *has at least two positive solutions* u_1, u_2 *with* $u_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

($H8$) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3$ *such that*

$$H_{\gamma\rho_1}^{\rho_1} > \rho_1 D^*,\ F_0^{\rho_2} < \rho_2 D,\ I_0^{\rho_2}(k) < \rho_2 D,\ J_0^{\rho_2}(k) < \rho_2 D,\ H_{\gamma\rho_3}^{\rho_3} > \rho_3 D^*.$$

Then problem (1.1) *has at least two positive solutions* u_1, u_2 *with* $u_1 \in K_{\rho_2} \setminus \overline{\Omega}_{\rho_1}$, $u_2 \in \Omega_{\rho_3} \setminus \overline{K}_{\rho_2}$.

Proof. We only consider the condition ($H7$). If ($H8$) holds, then the proof is similar to that of the case when ($H7$) holds. By Lemma 2.5, we know that the operator $T : K \to K$ is completely continuous.

First, we show that $i_K(T, K_{\rho_1}) = 1$. In fact, by (2.14), $F_0^{\rho_1} < \rho_1 D$, $I_0^{\rho_1}(k) < \rho_1 D$, $J_0^{\rho_1}(k) < \rho_1 D$, we have for $u \in \partial K_{\rho_1}$,

$$\begin{aligned}(Tu)(t) &\le \left(\int_0^1 G(\sigma(s), \sigma(s))\, q(s)\Delta s + \frac{n}{\rho}(c+d)(2a+b) + A(b+a) + B(d+c)\right)\rho_1 D \\ &= \rho_1,\end{aligned}$$

i.e., $\|Tu\| < \|u\|$ for $u \in \partial K_{\rho_1}$. By ($i$) of Lemma 3.1, we obtain that $i_K(T, K_{\rho_1}) = 1$.

Secondly, we show that $i_K(T, \Omega_{\rho_2}) = 0$. Let $e(t) \equiv 1$. Then $e \in \partial K_1$. We claim that

$$u \ne Tu + \lambda e,\ u \in \partial\Omega_{\rho_2},\ \lambda > 0.$$

Suppose that there exists $u_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 > 0$ such that

$$u_0 = Tu_0 + \lambda_0 e. \tag{3.1}$$

Then, Lemma 2.4, Lemma 3.2 and (3.1) imply that for $t \in [0, 1]_{\mathbb{T}}$

$$\begin{aligned}u_0 &= Tu_0 + \lambda_0 e \ge \gamma\|Tu_0\| + \lambda_0 \\ &\ge \gamma \min_{t\in[0,1]_{\mathbb{T}}}\left(\int_0^1 G(t, \sigma(s))\, q(s) f(s, u(s))\Delta s\right) + \lambda_0 \\ &> \gamma\left(\frac{d}{c+d}\int_0^1 G(\sigma(s), \sigma(s))\, q(s)\Delta s\right)\rho_2 D^* + \lambda_0 \\ &= \gamma\rho_2 + \lambda_0,\end{aligned}$$

i.e. $\gamma\rho_2 > \gamma\rho_2 + \lambda_0$, which is a contradiction. Hence by (ii) of Lemma 3.1, it follows that $i_K(T, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_K(T, K_{\rho_1}) = 1$, we can prove that $i_K(T, K_{\rho_3}) = 1$. Since $\rho_1 < \gamma\rho_2$ and Lemma 3.2 (b), we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. Similarly with $\rho_2 < \rho_3$ and Lemma 3.2 (b), we have $\overline{\Omega}_{\rho_2} \subset K_{\rho_2} \subset K_{\rho_3}$. Therefore ($iii$) of Lemma 3.1 implies that BVP (1.1) has at least two positive solutions u_1, u_2 with $u_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. □

Theorem 3.3 can be generalized to obtain many solutions.

Theorem 3.4. *Suppose* ($H1$) − ($H6$) *hold. Then we have the following assertions.*

($H9$) *There exists* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0, \infty)$ *with* $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \ldots < \gamma\rho_{2m_0} < \rho_{2m_0} < \rho_{2m_0+1}$ *such that*

$$F_0^{\rho_{2m-1}} < \rho_{2m-1} D,\ I_0^{\rho_{2m-1}}(k) < \rho_{2m-1} D,\ J_0^{\rho_{2m-1}}(k) < \rho_{2m-1} D,\ (m = 1, 2, \ldots, m_0, m_0 + 1),$$
$$H_{\gamma\rho_{2m}}^{\rho_{2m}} > \rho_{2m} D^*,\ (m = 1, 2, \ldots, m_0).$$

Then problem (1.1) *has at least* $2m_0$ *solutions in K.*

($H10$) *There exists* $\{\rho_i\}_{i=1}^{2m_0} \subset (0, \infty)$ *with* $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \ldots < \gamma\rho_{2m_0} < \rho_{2m_0}$ *such that*

$$F_0^{\rho_{2m-1}} < \rho_{2m-1} D,\ I_0^{\rho_{2m-1}}(k) < \rho_{2m-1} D,\ J_0^{\rho_{2m-1}}(k) < \rho_{2m-1} D,\ H_{\gamma\rho_{2m}}^{\rho_{2m}} > \rho_{2m} D^*,\ (m = 1, 2, \ldots, m_0).$$

Then problem (1.1) *has at least* $2m_0 - 1$ *solutions in K.*

Theorem 3.5. *Suppose* $(H1) - (H6)$ *hold. Then we have the following assertions.*

$(H11)$ *There exists* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < ... < \rho_{2m_0} < \gamma\rho_{2m_0+1} < \rho_{2m_0+1}$ *such that*

$$H_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \rho_{2m-1}D^*,\ (m = 1,2,...,m_0,m_0+1),\ F_0^{\rho_{2m}} < \rho_{2m}D,\ I_0^{\rho_{2m}}(k) < \rho_{2m}D,$$
$$J_0^{\rho_{2m}}(k) < \rho_{2m}D,\ (m = 1,2,...,m_0).$$

Then problem (1.1) *has at least* $2m_0$ *solutions in K.*

$(H12)$ *There exists* $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < ... < \gamma\rho_{2m_0-1} < \rho_{2m_0-1} < \rho_{2m_0}$ *such that*

$$H_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \rho_{2m-1}D^*,\ F_0^{\rho_{2m}} < \rho_{2m}D,\ I_0^{\rho_{2m}}(k) < \rho_{2m}D,\ J_0^{\rho_{2m}}(k) < \rho_{2m}D,\ (m = 1,2,...,m_0).$$

Then problem (1.1) *has at least* $2m_0 - 1$ *solutions in K.*

4. An Example

Example 4.1 In BVP (1.1), suppose that $\mathbb{T} = [0,1]$, $t_1 = \frac{1}{3}$, $q(t) = 1$, $a = b = c = 1$, $d = 2$, $\xi_1 = \alpha_1 = \frac{1}{2}$ and $\beta_1 = \frac{3}{2}$, i.e.,

$$\begin{cases} u''(t) + f(t,u(t)) = 0,\ t \in [0,1],\ t \neq \frac{1}{3}, \\ \Delta u|_{t=\frac{1}{3}} = I_1\left(u\left(\frac{1}{3}\right)\right), \\ \Delta u'|_{t=\frac{1}{3}} = -J_1\left(u\left(\frac{1}{3}\right)\right), \\ u(0) - u'(0) = \frac{1}{2}u\left(\frac{1}{2}\right), \\ u(1) + 2u'(1) = \frac{3}{2}u\left(\frac{1}{2}\right), \end{cases} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \frac{2}{75}u, & u \in [0,3], \\ \frac{498}{25}u - \frac{1492}{25}, & u \in (3,4], \\ \frac{5}{498}u + \frac{4970}{249}, & u \in (4,\infty), \end{cases}$$

$$I_1(u) = J_1(u) = \frac{748}{24925}u - \frac{10}{997},\ u \geq 0.$$

By simple calculation, we get $\rho = 4$, $\theta(t) = 1 + t$, $\varphi(t) = 3 - t$, $\Delta = -2$, $\gamma = \frac{1}{3}$, $A = \frac{147}{16}$, $B = \frac{49}{16}$, $D = \frac{48}{1475}$, $D^* = \frac{18}{11}$ and

$$G(t,s) = \frac{1}{4}\begin{cases} (1+s)(3-t), & s \leq t, \\ (1+t)(3-s), & t \leq s. \end{cases}$$

It is clear that $(H1) - (H6)$ are satisfied. Taking $\rho_1 = 3,\ \rho_2 = 12,\ \rho_3 = 1000$, we can obtain that

$$\rho_1 < \gamma\rho_2 \quad \text{and} \quad \rho_2 < \rho_3.$$

Now, we show that $(H7)$ is satisfied:

$$\begin{aligned}
&F_0^3 = 0.08 < \rho_1 D = 0.0976271,\ \ I_0^3 = J_0^3 = 0.08 < \rho_1 D = 0.0976271,\\
&H_4^{12} = 20 > \rho_2 D^* = 19.636363,\\
&F_0^{1000} = 30 < \rho_3 D = 32.54237288,\ \ I_0^{1000} = J_0^{1000} = 30 < \rho_3 D = 32.54237288.
\end{aligned}$$

Then, $(H7)$ condition of Theorem 3.3 hold. Hence, we get the BVP (4.1) has at least two positive solutions.

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